OCR
RECOGNISING ACHIEVEMENT

OXFORD CAMBRIDGE AND RSA EXAMINATIONS

Advanced Subsidiary General Certificate of Education
Advanced General Certificate of Education

MATHEMATICS 4726

Further Pure Mathematics 2

Specimen Paper

Additional materials:
Answer booklet
Graph paper
List of Formulae (MF 1)

TIME 1 hour 30 minutes

INSTRUCTIONS TO CANDIDATES

- Write your Name, Centre Number and Candidate Number in the spaces provided on the answer booklet.
- Answer **all** the questions.
- Give non-exact numerical answers correct to 3 significant figures, unless a different degree of accuracy is specified in the question or is clearly appropriate.
- You are permitted to use a graphic calculator in this paper.

INFORMATION FOR CANDIDATES

- The number of marks is given in brackets [ ] at the end of each question or part question.
- The total number of marks for this paper is 72.
- Questions carrying smaller numbers of marks are printed earlier in the paper, and questions carrying larger numbers of marks later in the paper.
- **You are reminded of the need for clear presentation in your answers.**

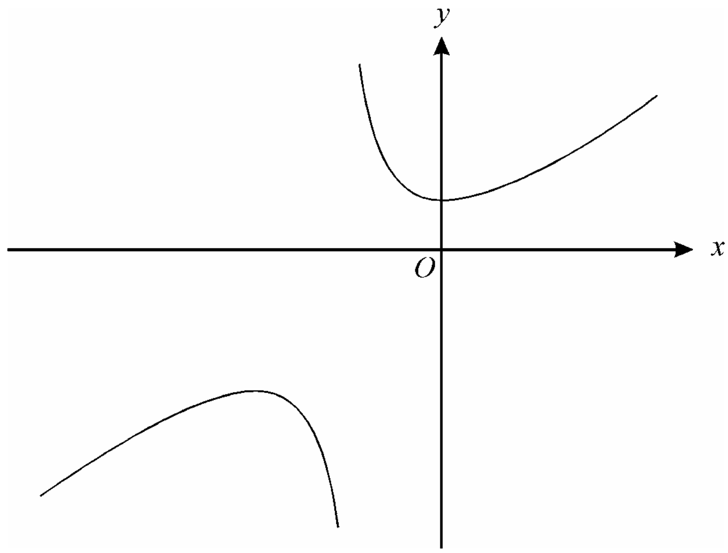
This question paper consists of 4 printed pages.

[Turn over

1 **(i)** Starting from the definition of $\cosh x$ in terms of e^x, show that $\cosh 2x = 2\cosh^2 x - 1$. [2]

(ii) Given that $\cosh 2x = k$, where $k > 1$, express each of $\cosh x$ and $\sinh x$ in terms of k. [4]

2



The diagram shows the graph of

$$y = \frac{2x^2 + 3x + 3}{x + 1}.$$

(i) Find the equations of the asymptotes of the curve. [3]

(ii) Prove that the values of y between which there are no points on the curve are -5 and 3. [4]

3 **(i)** Find the first three terms of the Maclaurin series for $\ln(2 + x)$. [4]

(ii) Write down the first three terms of the series for $\ln(2 - x)$, and hence show that, if x is small, then

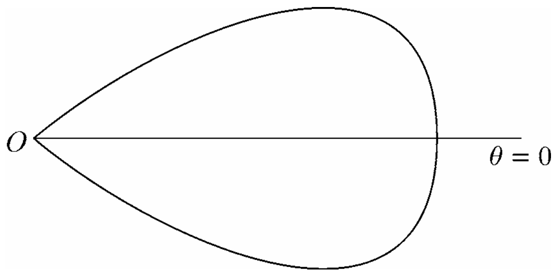
$$\ln\left(\frac{2+x}{2-x}\right) \approx x.$$ [3]

4 The equation of a curve, in polar coordinates, is

$$r = 2\cos 2\theta \qquad (-\pi < \theta \leqslant \pi).$$

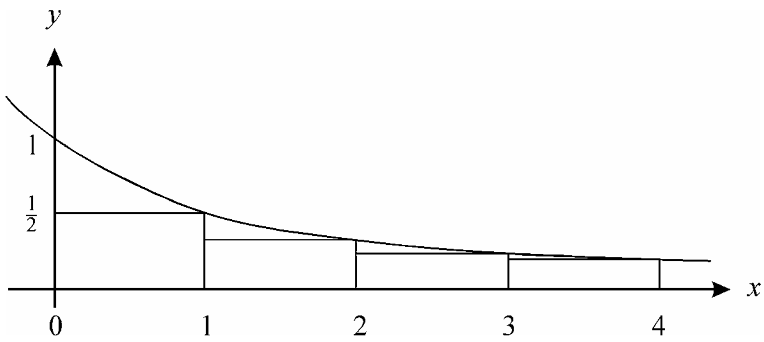
(i) Find the values of θ which give the directions of the tangents at the pole. [3]

One loop of the curve is shown in the diagram.



(ii) Find the exact value of the area of the region enclosed by the loop. [5]

5



The diagram shows the curve $y = \dfrac{1}{x+1}$ together with four rectangles of unit width.

(i) Explain how the diagram shows that

$$\tfrac{1}{2} + \tfrac{1}{3} + \tfrac{1}{4} + \tfrac{1}{5} < \int_0^4 \frac{1}{x+1}\,dx.$$ [2]

The curve $y = \dfrac{1}{x+2}$ passes through the top left-hand corner of each of the four rectangles shown.

(ii) By considering the rectangles in relation to this curve, write down a second inequality involving $\tfrac{1}{2} + \tfrac{1}{3} + \tfrac{1}{4} + \tfrac{1}{5}$ and a definite integral. [2]

(iii) By considering a suitable range of integration and corresponding rectangles, show that

$$\ln(500.5) < \sum_{r=2}^{1000} \frac{1}{r} < \ln(1000).$$ [4]

[Turn over

6 **(i)** Given that $I_n = \int_0^1 x^n \sqrt{(1-x)}\ \mathrm{d}x$, prove that, for $n \geqslant 1$,

$$(2n+3)I_n = 2nI_{n-1}.$$ [6]

(ii) Hence find the exact value of I_2. [4]

7 The curve with equation

$$y = \frac{x}{\cosh x}$$

has one stationary point for $x > 0$.

(i) Show that the x-coordinate of this stationary point satisfies the equation $x \tanh x - 1 = 0$. [2]

The positive root of the equation $x \tanh x - 1 = 0$ is denoted by α.

(ii) Draw a sketch showing (for positive values of x) the graph of $y = \tanh x$ and its asymptote, and the graph of $y = \frac{1}{x}$. Explain how you can deduce from your sketch that $\alpha > 1$. [3]

(iii) Use the Newton-Raphson method, taking first approximation $x_1 = 1$, to find further approximations x_2 and x_3 for α. [5]

(iv) By considering the approximate errors in x_1 and x_2, estimate the error in x_3. [3]

8 **(i)** Use the substitution $t = \tan \frac{1}{2}x$ to show that

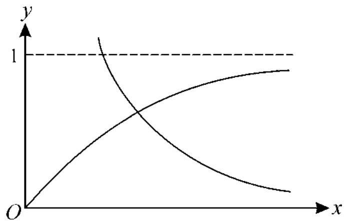
$$\int_0^{\frac{1}{2}\pi} \sqrt{\frac{1-\cos x}{1+\sin x}}\ \mathrm{d}x = 2\sqrt{2}\int_0^1 \frac{t}{(1+t)(1+t^2)}\ \mathrm{d}t.$$ [4]

(ii) Express $\frac{t}{(1+t)(1+t^2)}$ in partial fractions. [5]

(iii) Hence find $\int_0^{\frac{1}{2}\pi} \sqrt{\frac{1-\cos x}{1+\sin x}}\ \mathrm{d}x$, expressing your answer in an exact form. [4]

1	(i)	RHS $= 2\left(\frac{1}{2}(e^x + e^{-x})\right)^2 - 1 = \frac{1}{2}(e^{2x} + e^{-2x}) =$ LHS	M1		For correct squaring of $(e^x + e^{-x})$
			A1	2	For completely correct proof
	(ii)	$2\cosh^2 x - 1 = k \Rightarrow \cosh x = \sqrt{\left(\frac{1}{2}(1+k)\right)}$	M1		For use of **(i)** and solving for $\cosh x$
			A1		For correct positive square root only
		$2\sinh^2 x + 1 = k \Rightarrow \sinh x = \pm\sqrt{\left(\frac{1}{2}(k-1)\right)}$	M1		For use of $\cosh^2 x - \sinh^2 x = 1$, or equivalent
			A1	4	For both correct square roots
				6	
2	(i)	$x = -1$ is an asymptote	B1		For correct equation of vertical asymptote
		$y = 2x + 1 + \dfrac{2}{x+1}$	M1		For algebraic division, or equivalent
		Hence $y = 2x + 1$ is an asymptote	A1	3	For correct equation of oblique asymptote
	(ii)	*EITHER*: Quadratic $2x^2 + (3-y)x + (3-y) = 0$	M1		For using discriminant of relevant quadratic
		has no real roots if $(3-y)^2 < 8(3-y)$	A1		For correct inequality or equation in y
		Hence $(3-y)(-5-y) < 0$	M1		For factorising, or equivalent
		So required values are 3 and -5	A1		For given answer correctly shown
		OR: $\dfrac{dy}{dx} = 2 - \dfrac{2}{(x+1)^2} = 0$	M1		For differentiating and equating to zero
		Hence $(x+1)^2 = 1$	A1		For correct simplified quadratic in x
		So $x = -2$ and $0 \Rightarrow y = -5$ and 3	M1		For solving for x and substituting to find y
			A1	4	For given answer correctly shown
				7	
3	(i)	*EITHER*: If $f(x) = \ln(x+2)$, then $f'(x) = \dfrac{1}{2+x}$	M1		For at least one differentiation attempt
		and $f''(x) = -\dfrac{1}{(2+x)^2}$	A1		For correct first and second derivatives
		$f(0) = \ln 2, f'(0) = \frac{1}{2}, f''(0) = -\frac{1}{4}$	A1√		For all three evaluations correct
		Hence $\ln(x+2) = \ln 2 + \frac{1}{2}x - \frac{1}{8}x^2 + ...$	A1		For three correct terms
		OR: $\ln(2+x) = \ln[2(1+\frac{1}{2}x)]$	M1		For factorising in this way
		$= \ln 2 + \ln(1+\frac{1}{2}x)$	A1		For using relevant log law correctly
		$= \ln 2 + \frac{1}{2}x - \dfrac{(\frac{1}{2}x)^2}{2} + ...$	M1		For use of standard series expansion
		$= \ln 2 + \frac{1}{2}x - \frac{1}{8}x^2 + ...$	A1	4	For three correct terms
	(ii)	$\ln(2-x) \approx \ln 2 - \frac{1}{2}x - \frac{1}{8}x^2$	B1√		For replacing x by $-x$
		$\ln\left(\dfrac{2+x}{2-x}\right) \approx (\ln 2 + \frac{1}{2}x - \frac{1}{8}x^2) - (\ln 2 - \frac{1}{2}x - \frac{1}{8}x^2)$	M1		For subtracting the two series
		$\approx x$, as required	A1	3	For showing given answer correctly
				7	

4 (i)	$r=0 \Rightarrow \cos 2\theta = 0 \Rightarrow \theta = \pm\frac{1}{4}\pi, \pm\frac{3}{4}\pi$	M1 A1 A1 **3**	For equating r to zero and solving for θ For any two correct values For all four correct values and no others
(ii)	Area is $\frac{1}{2}\int_{-\frac{1}{4}\pi}^{\frac{1}{4}\pi} 4\cos^2 2\theta \, d\theta$ i.e. $\int_{-\frac{1}{4}\pi}^{\frac{1}{4}\pi} 1+\cos 4\theta \, d\theta = \left[\theta + \frac{1}{4}\sin 4\theta\right]_{-\frac{1}{4}\pi}^{\frac{1}{4}\pi} = \frac{1}{2}\pi$	M1 B1√ M1 A1 A1 **5**	For us of correct formula $\frac{1}{2}\int r^2 \, d\theta$ For correct limits from **(i)** For using double-angle formula For $\theta + \frac{1}{4}\sin 4\theta$ correct For correct (exact) answer
		8	
5 (i)	LHS is the total area of the four rectangles RHS is the corresponding area under the curve, which is clearly greater	B1 B1 **2**	For identifying rectangle areas (not heights) For correct explanation
(ii)	$\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5} > \int_0^4 \frac{1}{x+2} \, dx$	M1 A1 **2**	For attempt at relevant new inequality For correct statement
(iii)	Sum is the area of 999 rectangles Bounds are $\int_0^{999} \frac{1}{x+2} \, dx$ and $\int_0^{999} \frac{1}{x+1} \, dx$ So lower bound is $[\ln(x+2)]_0^{999} = \ln(500.5)$ and upper bound is $[\ln(x+1)]_0^{999} = \ln(1000)$	M1 M1 A1 A1 **4**	For considering the sum as an area again For stating either integral as a bound For showing the given value correctly Ditto
		8	
6 (i)	$I_n = \left[-\frac{2}{3}x^n(1-x)^{\frac{3}{2}}\right]_0^1 + \frac{2}{3}n\int_0^1 x^{n-1}(1-x)^{\frac{3}{2}} \, dx$ $= \frac{2}{3}n\int_0^1 x^{n-1}(1-x)\sqrt{(1-x)} \, dx$ $= \frac{2}{3}n(I_{n-1} - I_n)$ Hence $(2n+3)I_n = 2nI_{n-1}$, as required	M1 A1 M1 M1 A1 A1 **6**	For using integration by parts For correct first stage result For use of limits in integrated term For splitting the remaining integral up For correct relation between I_n and I_{n-1} For showing given answer correctly
(ii)	$I_2 = \frac{4}{7}I_1 = \frac{4}{7}\times\frac{2}{5}I_0$ Hence $I_2 = \frac{8}{35}\left[-\frac{2}{3}(1-x)^{\frac{3}{2}}\right]_0^1 = \frac{16}{105}$	M1 A1 M1 A1 **4**	For two uses of the recurrence relation For correct expression in terms of I_0 For evaluation of I_0 For correct answer
		10	

7 **(i)** $\frac{dy}{dx} = \frac{\cosh x - x\sinh x}{\cosh^2 x}$	M1	For differentiating and equating to zero
Max occurs when $\cosh x = x\sinh x$, i.e. $x\tanh x = 1$	A1 **2**	For showing given result correctly
(ii) 	B1	For correct sketch of $y = \tanh x$
	B1	For identification of asymptote $y = 1$
	B1 **3**	For correct explanation of $\alpha > 1$ based on intersection $(1, 1)$ of $y = 1/x$ with $y = 1$
(iii) $x_{n+1} = x_n - \frac{x_n \tanh x_n - 1}{\tanh x_n + x_n \operatorname{sech}^2 x_n}$	M1	For correct Newton-Raphson structure
	A1	For all details in $x - \frac{f(x)}{f'(x)}$ correct
$x_1 = 1 \Rightarrow x_2 = 1.20177...$	M1	For using Newton-Raphson at least once
	A1	For x_2 correct to at least 3sf
$x_3 = 1.1996785...$	A1 **5**	For x_3 correct to at least 4sf
(iv) $e_1 \approx 0.2,\ e_2 \approx -0.002$	B1√	For both magnitudes correct
$\frac{e_3}{e_2^2} \approx \frac{e_2}{e_1^2} \Rightarrow e_3 \approx -2\times10^{-7}$	M1	For use of quadratic convergence property
	A1 **3**	For answer of correct magnitude
	13	
8 **(i)** $\frac{dt}{dx} = \frac{1}{2}(1+t^2)$	B1	For this relation, stated or used
$\int_0^{\frac{1}{2}\pi} \sqrt{\frac{1-\cos}{1+\sin x}}\, dx = \int_0^1 \sqrt{\frac{1-\frac{1-t^2}{1+t^2}}{1+\frac{2t}{1+t^2}}} . \frac{2}{1+t^2}\, dt$	M1	For complete substitution for x in integrand
	B1	For justification of limits 0 and 1 for t
$= \int_0^1 \sqrt{\frac{2t^2}{(1+t)^2}} . \frac{2}{1+t^2}\, dt = 2\sqrt{2}\int_0^1 \frac{t}{(1+t)(1+t^2)}\, dt$	A1 **4**	For correct simplification to given answer
(ii) $\frac{t}{(1+t)(1+t^2)} = \frac{A}{1+t} + \frac{Bt + C}{1+t^2}$	B1	For statement of correct form of pfs
Hence $t \equiv A(1+t^2) + (Bt + C)(1+t)$	M1	For any use of the identity involving B or C
From which $A = -\frac{1}{2}, B = \frac{1}{2}, C = \frac{1}{2}$	B1	For correct value of A
	A1	For correct value of B
	A1 **5**	For correct value of C
(iii) Int is $2\sqrt{2}\left[-\frac{1}{2}\ln(1+t) + \frac{1}{4}\ln(1+t^2) + \frac{1}{2}\tan^{-1} t\right]_0^1$	B1√	For both logarithm terms correct
	B1√	For the inverse tan term correct
$= \frac{1}{4}(\pi - 2\ln 2)\sqrt{2}$	M1	For use of appropriate limits
	A1 **4**	For correct (exact) answer in any form
	13	